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Lebesgue-Stieltjes Integrablity of x^n with Respect to Unbounded Monotone Functions

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1. Introduction.

B. Sz.-Nagy [1] showed the following theorem.

Theorem A. If $0 < \gamma \leqq 1$, $f(x)$ *decreases on* $(0, \pi)$, $f(\pi - 0) > -\infty$ *and* $xf(x) \in L(0, \pi)$, *then* $x^{\gamma-1} f(x) \in L(0, \pi)$ *if and only if* $\sum n^{-\gamma} |b_n| < \infty$ *where* $b_n = 2\pi^{-1} \int_0^\pi f(x) \sin nx \, dx$.

It is easy to check that the statement of Theorem A for $1 < \gamma < 2$ is still true. But when $\gamma = 0$, the theorem fails; as an example we may take $f(x) = -x$, then $x^{-1} f(x) \in L(0, \pi)$, $|b_n| = 2n^{-1}$ and $\sum |b_n| = \infty$. Recently as a replacement for the case $\gamma = 0$ of Theorem A, R. P. Boas [2] showed

Theorem B. If $f(x)$ *decreases on* $(0, \pi)$ *and* $\int_{(0,\pi)} x^2 |df(x)| < \infty$, *then* $f(x)$ *is bounded if and only if* $n^{-1} \sum_{k=1}^{n} kb_k = O(1)$ *where* $b_k = 2\pi^{-1} \int_0^\pi f(x) \sin kx \, dx$.

Moreover he [2] gave the following two theorems.

Theorem B′. If $f(x)$ *decreases on* $(0, \pi)$ *and* $\int_{(0,\pi)} x^2 |df(x)| < \infty$, *then* $f(x)$ *is bounded if and only if* $n^{-1} \sum_{k=1}^{n} a_k = O(1)$ *where* $a_k = -2\pi^{-1} \int_{(0,\pi)} (1 - \cos kx) \, df(x)$.

Theoeem C. If $g(x) \geqq 0$ *on* $(0, \pi)$ *and* $x^3 g(x) \in L(0, \pi)$, *then* $xg(x) \in L(0, \pi)$ *if and only if* $n^{-1} \sum_{k=1}^{n} k^{-1} b_k = O(1)$ *where* $b_k = -2\pi^{-1} \int_0^\pi (kx - \sin kx) \, g(x) \, dx$.

It is evident by Nagy's lemma ([1], p. 119) that Theorem B′ is equivalent to Theorem B. The aim of this paper is to give a generalization of these results being due to Boas.

2. Theorems.

Theorem 1. *Let m be a non-negative integer and* $f(x)$ *be af unction of boun-*

ded variation on $[\varepsilon, \pi]$ *for every* $\varepsilon > 0$. (i) *If*

$$\int_{(0,\pi)} x^{2m} \, | df(x) | < \infty, \tag{2.1}$$

then we have

$$\frac{1}{n} \sum_{k=1}^{n} \frac{| a_k |}{k^{2m}} = O(1), \tag{2.2}$$

where

$$a_k = \frac{2}{\pi} \int_{(0,\pi)} \left\{ \cos kx - \sum_{j=0}^{m} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right\} df(x) \quad k = 1, 2, \cdots, \tag{2.3}$$

(ii) *If*

$$\int_{(0,\pi)} x^{2m+1} \, | df(x) | < \infty, \tag{2.4}$$

then we have

$$\frac{1}{n} \sum_{k=1}^{n} \frac{| b_k |}{k^{2m+1}} = O(1), \tag{2.5}$$

where

$$b_k = \frac{2}{\pi} \int_{(0,\pi)} \left\{ \sin kx - \sum_{j=0}^{m} (-1)^j \frac{(kx)^{2j+1}}{(2j+1)!} \right\} df(x) \qquad k = 1, 2, \cdots, \tag{2.6}$$

Theorem 2. *Let* m *be a non-negative integer and* $f(x)$ *decrease on* $(0, \pi)$. (i) *If*

$$\int_{(0,\pi)} x^{2m+2} \, | df(x) | < \infty \tag{2.7}$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{k^{2m}} = O(1), \tag{2.8}$$

where a_k'*s are defined by* (2.3), *then we have* (2.1). (ii) *If*

$$\int_{(0,\pi)} x^{2m+3} \, | df(x) | < \infty \tag{2.9}$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \frac{b_k}{k^{2m+1}} = O(1), \tag{2.10}$$

where b_k'*s are defined by* (2.6), *then we have* (2.4).

The cases $m = 0$ in Theorem 1 (i) and Theorem 2 (i) yield Theorem B′. Similarly the cases $m = 0$ in Theorem 1 (ii) and Theorem 2 (ii) yield Theorem C.

Now we prove Theorem 1 (i) and Theorem 2 (i). The proof for Theorem 1 (ii) and Theorem 2 (ii) will be proceeded quite similarly.

Proof of Theorem 1 (*i*).

By (2.1) and (2.3), we have

$$| a_k | \leqq \frac{2}{\pi} \int_{(0,\pi)} 2 \frac{(kx)^{2m}}{(2m)!} | df(x) |$$

$$= k^{2m} \frac{4}{\pi} \frac{1}{(2m)!} \int_{(0,\pi)} x^{2m} \, | df(x) | = C \, k^{2m}, \quad k = 1, 2, \cdots,$$

Hence

$$0 \leqq \frac{1}{n} \sum_{k=1}^{n} \frac{|a_k|}{k^{2m}} \leqq C, \qquad n = 1, 2, \cdots.$$

Proof of Theorem 2 (i).

First of all let $m = 0$. Then

$$a_k = \frac{2}{\pi} \int_{(0,\pi)} (1 - \cos kx) \, | df(x) | \geqq 0, \qquad k = 1, 2, \cdots,$$

$$0 \leqq \frac{1}{n} \sum_{k=1}^{n} a_k \leqq C, \qquad n = 1, 2, \cdots,$$

and

$$\frac{\pi}{2} \frac{1}{n} \sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{n} \int_{(0,\pi)} (1 - \cos kx) \, | df(x) |$$

$$= \int_{(0,\pi)} \left\{ 1 - \frac{1}{n} \frac{\sin (n + \frac{1}{2}) x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} \right\} | df(x) |.$$

Letting $n \to \infty$, we have $\frac{\pi}{2} C \geqq \int_{(0,\pi)} | df(x) |$ by Fatou's lemma.

Secondly let $m \geqq 1$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{k^{2m}} \right| \leqq C, \qquad n = 1, 2, \cdots,$$

$$\frac{\pi}{2} \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{k^{2m}} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k^{2m}} \int_{(0,\pi)} \left\{ \cos kx - \sum_{j=0}^{m} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right\} df(x)$$

$$= \int_{(0,\pi)} \left\{ \frac{(-1)^m}{(2m)!} x^{2m} - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k^{2m}} (\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!}) \right\} | df(x) |.$$

Therefore

$$(-1)^m \frac{\pi}{2} \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{k^{2m}}$$

$$= \int_{(0,\pi)} \left\{ \frac{x^{2m}}{(2m)!} - \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^m}{k2^m} (\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right. | df(x) |.$$

And we know

$$\frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^m}{k^{2m}} (\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!})$$

$$= \frac{1}{n} (-1)^m \sum_{k=1}^{n} \frac{\cos kx}{k^{2m}} - \sum_{j=0}^{mm1} \frac{(-1)^{m+j} x^{2j}}{(2j)!} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k^{2(m-j)}} \to 0 \qquad (n \to \infty)$$

and

$$\frac{x^{2m}}{(2m)!} - \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^m}{k^{2m}} (\cos kx - \sum_{j=0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!}) \geqq 0$$

because

$$\left| \cos kx - \sum_{j-0}^{m-1} (-1)^j \frac{(kx)^{2j}}{(2j)!} \right| \leqq \frac{(kx)^{2m}}{(2m)!} .$$

Hence by Fatou's lemma

$$\frac{\pi}{2}C \geqq \frac{1}{(2m)!}\int_{(0,\pi)} x^{2m}\,|\,df(x)|.$$

References

[1] B. SZ.-NAGY, Séries et intégrales de Fourier des fonctions monotones non bornées, Acta Sci. Math. 13 (1949), 118–135.

[2] R. P. BOAS, Jr., Integrability of non-negative trigonometric series II, Tôhoku Math. Journ. 16 (1964), 368–373.